Internal Indecomposability of Profinite Groups

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Abstract

It is well-known that various profinite groups appearing in anabelian geometry — e.g., the absolute Galois groups of p-adic local fields or number fields — satisfy distinctive group-theoretic properties such as *slimness* [i.e., the property that every open subgroup is center-free] and *strong indecomposability* [i.e., the property that every open subgroup has no nontrivial product decomposition]. In the present paper, we consider another group-theoretic property on profinite groups, which we shall refer to as *strong internal indecomposability.* This is a *stronger* property than *both* slimness and strong indecomposability. In the present paper, we examine basic properties of strong internal indecomposability and prove that the absolute Galois groups of Henselian discrete valuation fields with positive characteristic residue fields or Hilbertian fields [which may be regarded as generalizations of p-adic local fields or number fields] satisfy strong internal indecomposability.

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Contents

Introduction

For any field F, we shall write F^{sep} for the separable closure [determined up to isomorphisms] of F; $G_F \stackrel{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$. Let p be a prime number.

Let X be an algebraic variety [i.e., a separated, of finite type, and geometrically integral scheme] over a field. In anabelian geometry, we often consider

> *whether or not the algebraic variety X may be "reconstructed" from the étale fundamental group $\pi_1(X)$.*

For instance, if X is a hyperbolic curve over a p-adic local field [i.e., a finite extension field of the field of p-adic numbers] or a number field [i.e., a finite extension field of the field of rational numbers], then Mochizuki and Tamagawa proved that X may be "reconstructed" from $\pi_1(X)$ [cf. [9], Theorem A; [10], Introduction; [15], Theorem 0.4]. However, it seems far-reaching to specify the precise class of algebraic varieties which may be "reconstructed" from their étale fundamental groups [i.e., the class of "anabelian varieties"].

On the other hand, it has been observed that various profinite groups appearing in anabelian geometry [e.g., the absolute Galois groups of p-adic local fields or number fields; the étale fundamental groups of hyperbolic curves over p-adic local fields or number fields] tend to satisfy group-theoretic properties such as *slimness* and *strong indecomposability* [cf. [7], [8]]. For our purposes, let us recall the definitions of the slimness and the strong indecomposability of profinite groups. Let G be a profinite group. Then we shall say that:

- G is *slim* if every open subgroup of G is center-free.
- G is *strongly indecomposable* if every open subgroup of G is indecomposable, i.e., has no nontrivial product decomposition.

However, at the time of writing the present paper, the authors do not know the precise relation between the class of "anabelian varieties" and the class of algebraic varieties that satisfy the above group-theoretic properties. It seems to the authors that a further examination of this relation would be important.

In this context, it is natural to pose the following question:

> Question 1: Do various profinite groups appearing in anabelian geometry satisfy stronger properties than slimness and strong indecomposability?

With regard to Question 1, in the present paper, we consider the notion of *strong internal indecomposability*, which is a stronger property than both slimness and strong indecomposability. Let $H \subseteq G$ be a normal closed subgroup. Then we shall say that:

- H is *normally decomposable* in G if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $H = H_1 \times H_2$.
- H is *normally indecomposable* in G if H is not normally decomposable in G.

- G is *internally indecomposable* if every normal closed subgroup of G is center-free and normally indecomposable in G.
- G is *strongly internally indecomposable* if every open subgroup of G is internally indecomposable.

Note that, if G is strongly internally indecomposable, then it follows immediately from the various definitions involved that G is slim and strongly indecomposable. Moreover, we also note that

> G is internally indecomposable if and only if, for every nontrivial normal closed subgroup $J \subseteq G$, the centralizer of J in G is trivial [cf. Proposition 1.2].

In anabelian geometry, this latter property has been considered and proved for special "$J \subseteq G$" [cf. [5], Lemma 2.13, (ii); [11], Lemma 2.7, (vi)]. Thus, it would be important to establish generalities on this property. One notable advantage of internal indecomposability — compared to indecomposability — is

> to behave reasonably well with respect to taking *limits* and *group extensions* [cf. Propositions 1.8; 1.11].

Moreover, it would be also important to consider the following question:

> Question 2: What types of profinite groups do satisfy strong internal indecomposability?

With regard to Question 2, in the present paper, we focus on the case of the absolute Galois groups. By making use of the above advantage [together with results in [8] and the theory of fields of norms], we obtain the following theorem [cf. Theorems 2.3; 2.6; 2.12; Remark 2.12.1]:

Theorem A.

(i) Let K be a Henselian discrete valuation field of residue characteristic p. Then G_K is strongly internally indecomposable. Moreover, if K contains a primitive p-th root of unity in the case where K is of characteristic 0, then any almost pro-p-maximal quotient of G_K [cf. Definition 1.4] is strongly internally indecomposable.

(ii) Let K be a Hilbertian field [i.e., a field for which Hilbert's irreducibility theorem holds — cf. Remark 2.12.1]. Then G_K is strongly internally indecomposable.

Note that p-adic local fields (respectively, number fields) are Henselian discrete valuation fields of residue characteristic p (respectively, Hilbertian fields). Thus, Theorem A may be regarded as a generalization of the well-known fact that the absolute Galois groups of p-adic local fields or number fields are slim and strongly indecomposable.

In our subsequent papers, we will discuss

- the strong internal indecomposability of the étale fundamental groups of various algebraic varieties appearing in anabelian geometry, and
- some applications of internal indecomposability to anabelian geometry.

The present paper is organized as follows. In §1, we introduce the notion of internal indecomposability of profinite groups and examine some basic properties of this notion which will be of later use. In §2, by applying results obtained in §1 of the present paper and [8], we prove that the absolute Galois groups of Henselian discrete valuation fields with positive characteristic residue fields and Hilbertian fields are strongly internally indecomposable [cf. Theorem A].

Notations and conventions

Numbers: The notation $\mathbb{Q}$ will be used to denote the field of rational numbers. The notation $\mathbb{Z}$ will be used to denote the ring of integers. The notation $\widehat{\mathbb{Z}}$ will be used to denote the profinite completion of the underlying additive group of $\mathbb{Z}$. The notation $\mathbb{N}$ will be used to denote the set of positive integers. If p is a prime number, then the notation $\mathbb{Z}_p$ will be used to denote the ring of p-adic integers.

Fields: Let F be a field; F^{sep} a separable closure of F; p a prime number. Then we shall write $\mathrm{char}(F)$ for the characteristic of F; $G_F \overset{\mathrm{def}}{=} \mathrm{Gal}(F^{\mathrm{sep}}/F)$; $F((t))$ for the one parameter formal power series field over F; $F_{p^\infty} \subseteq F^{\mathrm{sep}}$ for the subfield obtained by adjoining p-power roots of unity to F. If $\mathrm{char}(F) \neq p$, then we shall fix a primitive p^i-th root of unity $\zeta_{p^i} \in F^{\mathrm{sep}}$ for each $i \in \mathbb{N}$. If F is perfect, then we shall also write $\overline{F} \overset{\mathrm{def}}{=} F^{\mathrm{sep}}$.

Profinite groups: Let G be a profinite group. We shall write $\mathrm{Aut}(G)$ for the group of continuous automorphisms of G; $\mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$ for the group of inner automorphisms of G; $\mathrm{Out}(G) \overset{\mathrm{def}}{=} \mathrm{Aut}(G)/\mathrm{Inn}(G)$. If p is a prime number, then we shall write G^p for the maximal pro-p quotient of G.

1 Basic properties of internal indecomposability

In the present section, we introduce the notion of *internal indecomposability* of profinite groups and examine basic properties.

Let p be a prime number.

Definition 1.1 ([10], Notations and Conventions; [10], Definition 1.1, (ii))**.** Let G be a profinite group; $H \subseteq G$ a closed subgroup of G.

(i) We shall write $Z_G(H)$ for the *centralizer* of H in G, i.e., the closed subgroup $\{g \in G \mid ghg^{-1} = h \text{ for any } h \in H\}$; $Z(G) \stackrel{\text{def}}{=} Z_G(G)$; $N_G(H)$ for the *normalizer* of H in G, i.e., the closed subgroup $\{g \in G \mid gHg^{-1} = H\}$.

(ii) We shall say that G is *slim* if $Z_G(U) = \{1\}$ for every open subgroup U of G. [Note that G is slim if and only if $Z(U) = \{1\}$ for every open subgroup U of G — cf. [8], Proposition 1.2, (i).]

(iii) We shall say that G is *decomposable* if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $G = H_1 \times H_2$. We shall say that G is *indecomposable* if G is not decomposable. We shall say that G is *strongly indecomposable* if every open subgroup of G is indecomposable.

(iv) We shall say that H is *normally decomposable* in G if there exist nontrivial normal closed subgroups $H_1 \subseteq G$ and $H_2 \subseteq G$ such that $H = H_1 \times H_2$. We shall say that H is *normally indecomposable* in G if H is not normally decomposable in G.

(v) We shall say that G is *internally indecomposable* if every nontrivial normal closed subgroup of G is center-free and normally indecomposable in G. [Note that the trivial subgroup of G is center-free and normally indecomposable in G.] We shall say that G is *strongly internally indecomposable* if every open subgroup of G is internally indecomposable.

Remark 1.1.1. It follows immediately from the various definitions involved that strongly internally indecomposable profinite groups are slim.

Remark 1.1.2. Let G be a profinite group. Then it follows immediately from the various definitions involved that:

(i) G is normally decomposable in G if and only if G is decomposable.

(ii) If G is internally indecomposable (respectively, strongly internally indecomposable), then G is indecomposable (respectively, strongly indecomposable).

Remark 1.1.3. Let G be a nonabelian finite simple group. Then it follows immediately from the various definitions involved that G is not strongly internally indecomposable but internally indecomposable.

Next, we give a useful criterion of internal indecomposability.

Proposition 1.2. *Let G be a profinite group. Then G is internally indecomposable if and only if $Z_G(H) = \{1\}$ for every nontrivial normal closed subgroup $H \subseteq G$.*

Proof. First, we verify *sufficiency.* Suppose that $Z_G(H) = \{1\}$ for every nontrivial normal closed subgroup $H \subseteq G$. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then $Z(H) \subseteq Z_G(H) = \{1\}$. On the other hand, let $H_1 \subseteq G$ and $H_2 \subseteq G$ be normal closed subgroups such that $H = H_1 \times H_2$, and $H_1 \neq \{1\}$. Then $H_2 \subseteq Z_G(H_1) = \{1\}$. Thus, we conclude that H is center-free and normally indecomposable in G, hence that G is internally indecomposable.

Next, we verify *necessity.* Suppose that G is internally indecomposable. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Since H is center-free, $H \cap Z_G(H) = Z(H) = \{1\}$. In particular, we obtain a normal closed subgroup $H \times Z_G(H) \subseteq G$. Thus, since G is internally indecomposable, and $H \neq \{1\}$, we conclude that $Z_G(H) = \{1\}$. This completes the proof of Proposition 1.2. □

Next, we recall basic notions concerning profinite groups.

Definition 1.3 ([12], Definition 1.1, (i), (ii))**.** Let $\mathcal{C}$ be a family of finite groups including the trivial group. Then:

(i) We shall refer to a finite group belonging to $\mathcal{C}$ as a $\mathcal{C}$-group.

(ii) We shall refer to $\mathcal{C}$ as a *full-formation* if $\mathcal{C}$ is closed under taking quotients, subgroups, and extensions.

(iii) We shall write $\Sigma_{\mathcal{C}}$ for the set of primes l such that $\mathbb{Z}/l\mathbb{Z}$ is a $\mathcal{C}$-group.

Definition 1.4 ([10], Definition 1.1, (iii))**.** Let G, Q be profinite groups; $q : G \twoheadrightarrow Q$ an epimorphism [in the category of profinite groups]. Then we shall say that Q is an *almost pro-p-maximal quotient* of G if there exists a normal open subgroup $N \subseteq G$ such that $\mathrm{Ker}(q)$ coincides with the kernel of the natural surjection $N \twoheadrightarrow N^p$.

Next, we recall the following result, which is one of the motivations of our research.

Proposition 1.5 ([14], Proposition 8.7.8)**.** *Let $\mathcal{C}$ be a full-formation; F a free pro-$\mathcal{C}$ group of rank ≥ 2. Then F is strongly internally indecomposable [cf. Proposition 1.2; [14], Theorem 3.6.2, (a)].*

Next, we recall an important property of slim profinite groups.

Lemma 1.6 ([8], Lemma 1.3; [12], §0, Topological Groups)**.** *Let G be a slim profinite group; $F \subseteq G$ a finite normal subgroup. Then $F = \{1\}$.*

In the following, we verify various important properties of internally indecomposable profinite groups.

Proposition 1.7. *Let G be a slim profinite group. Suppose that there exists an open subgroup $H \subseteq G$ such that H is internally indecomposable (respectively, strongly internally indecomposable). Then G is internally indecomposable (respectively, strongly internally indecomposable).*

Proof. First, observe that, for any open subgroup $U \subseteq G$ and any nontrivial normal closed subgroup $\Gamma \subseteq G$, it holds that $\Gamma \cap U \neq \{1\}$. Indeed, suppose that $\Gamma \cap U = \{1\}$. Then since $\Gamma \cap U$ is open in Γ, we conclude that Γ is finite. Thus, it follows from Lemma 1.6 that $\Gamma = \{1\}$. This is a contradiction. Therefore, we have $\Gamma \cap U \neq \{1\}$. This completes the proof of the observation.

Next, note that, to verify Proposition 1.7, it suffices to prove the non-resp'd case. Let $H \subseteq G$ be an internally indecomposable open subgroup; $N \subseteq G$ a nontrivial normal closed subgroup. Then it follows from the above observation that $N \cap H \neq \{1\}$. Write $C \stackrel{\text{def}}{=} Z_G(N)$. Note that since H is internally indecomposable, we have $C \cap H \subseteq Z_H(N \cap H) = \{1\}$ [cf. Proposition 1.2]. Again, it follows from the above observation that $C = \{1\}$. Thus, we conclude that G is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Proposition 1.7. $\square$

Proposition 1.8. *Let G be a profinite group; $\{G_i\}_{i \in I}$ a directed subset of the set of normal closed subgroups of G — where $j \geq i \Leftrightarrow G_j \subseteq G_i$ — such that the natural homomorphism*

$$G \to \varprojlim_{i \in I} G/G_i$$

is an isomorphism. Suppose that, for each $i \in I$, G/G_i is internally indecomposable (respectively, strongly internally indecomposable). Then G is internally indecomposable (respectively, strongly internally indecomposable).

Proof. To verify Proposition 1.8, it suffices to prove the non-resp'd case. For each $i \in I$, write $\phi_i : G \twoheadrightarrow G/G_i$ for the natural surjection. Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then since $G \xrightarrow{\sim} \varprojlim_{i \in I} G/G_i$, there exists $i \in I$ such that

$$\phi_i(H) \neq \{1\}.$$

Fix such $i \in I$. Write $I_i \stackrel{\text{def}}{=} \{j \in I \mid j \geq i\}$; $C \stackrel{\text{def}}{=} Z_G(H)$. Since $\{G_i\}_{i \in I}$ is a directed set, the natural homomorphism

$$G \to \varprojlim_{j \in I_i} G/G_j$$

is an isomorphism. Let $j \in I_i$ be an element. Observe that

- $\phi_j(H) \neq \{1\}$,
- $\phi_j(H)$ and $\phi_j(C)$ are normal closed subgroups of G/G_j, and
- $\phi_j(H) \subseteq Z_{G/G_j}(\phi_j(C))$.

Then since G/G_j is internally indecomposable, it holds that $\phi_j(C) = \{1\}$ [cf. Proposition 1.2]. Thus, it follows from the equality

$$\bigcap_{j \in I_i} G_j = \{1\}$$

that $C = \{1\}$, hence that G is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Proposition 1.8. □

Next, we give a variant of [7], Lemma 1.6.

Lemma 1.9. *Let G be an internally indecomposable profinite group; $H \subseteq G$ a nontrivial normal closed subgroup; $\alpha \in \mathrm{Aut}(G)$. Suppose that, for any $h \in H$, it holds that $\alpha(h) = h$. Then α is the identity automorphism.*

Proof. Let $g \in G$ be an element. Then, for any $h \in H$, we have

$$\begin{aligned} \alpha(g) \cdot g^{-1} \cdot h \cdot (\alpha(g) \cdot g^{-1})^{-1} &= \alpha(g) \cdot g^{-1} \cdot h \cdot g \cdot \alpha(g)^{-1} \\ &= \alpha(g) \cdot \alpha(g^{-1} \cdot h \cdot g) \cdot \alpha(g)^{-1} \\ &= \alpha(h) \\ &= h, \end{aligned}$$

where the second equality follows from the fact that $g^{-1} \cdot h \cdot g \in H$. Thus, it follows from Proposition 1.2 that $\alpha(g) \cdot g^{-1} \in Z_G(H) = \{1\}$. Therefore, we conclude that α is the identity automorphism. This completes the proof of Lemma 1.9. □

The following lemma will be applied in the proof of Proposition 1.11 below.

Lemma 1.10 ([7], Lemma 1.7, (i))**.** *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_1 & \longrightarrow & \Pi_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_2 & \longrightarrow & \Pi_2 & \longrightarrow & G_2 & \longrightarrow & 1 \end{array}$$

be a commutative diagram of profinite groups, where the horizontal sequences are exact; the vertical arrows are open injections. Write

$$\rho_1 : G_1 \to \mathrm{Out}(\Delta_1) \quad \textit{(respectively, } \rho_2 : G_2 \to \mathrm{Out}(\Delta_2)\textit{)}$$

for the natural outer representation associated to the upper (respectively, lower) horizontal sequence. Suppose that Δ_2 is slim. Then $\mathrm{Ker}(\rho_1)$ is an open subgroup of $\mathrm{Ker}(\rho_2)$.

Proposition 1.11. *Let*

$$1 \longrightarrow G_1 \longrightarrow G \longrightarrow G_2 \longrightarrow 1$$

be an exact sequence of profinite groups. Write $\rho : G_2 \to \mathrm{Out}(G_1)$ *for the outer representation associated to this exact sequence. Then the following hold:*

(i) Suppose that

- G_1 *is internally indecomposable (respectively, strongly internally indecomposable);*
- G_2 *is internally indecomposable (respectively, strongly internally indecomposable);*
- ρ *is injective.*

Then G *is internally indecomposable (respectively, strongly internally indecomposable).*

(ii) Suppose that

- G_1 *is internally indecomposable (respectively, strongly internally indecomposable);*
- G_2 *is abelian;*
- ρ *is injective, or* G *is center-free (respectively, slim).*

Then G *is internally indecomposable (respectively, strongly internally indecomposable).*

Proof. It follows immediately from Lemma 1.10, together with Remark 1.1.1, that, to verify Proposition 1.11, it suffices to prove the non-resp'd case. Let $N \subseteq G$ be a nontrivial normal closed subgroup. Write

$$C \stackrel{\text{def}}{=} Z_G(N), \quad C_1 \stackrel{\text{def}}{=} Z_{G_1}(N \cap G_1).$$

Our goal is to prove that $C = \{1\}$ [cf. Proposition 1.2].

First, we verify assertion (i). Let us begin by observing the following assertion:

> Claim 1.11.A: Let $H \subseteq G$ be a nontrivial normal closed subgroup. Suppose that $Z_G(H) \subseteq G_1$. Then $Z_G(H) = \{1\}$.

Indeed, suppose that $Z_G(H) \neq \{1\}$. Then since G_1 is internally indecomposable, and $Z_G(H) \subseteq G_1$ is normal, it holds that $H \subseteq Z_G(G_1)$ [cf. Lemma 1.9]. On the other hand, it follows immediately from our assumption that ρ is injective that $Z_G(G_1) \subseteq Z(G_1)$. Moreover, since G_1 is center-free, it holds that $Z_G(G_1) = \{1\}$, hence that $H = \{1\}$. This is a contradiction. Thus, we conclude that $Z_G(H) = \{1\}$. This completes the proof of Claim 1.11.A.

Suppose that $N \cap G_1 = \{1\}$. Then since $N \neq \{1\}$, and G_2 is internally indecomposable, it holds that $C \subseteq G_1$. Thus, by applying Claim 1.11.A to the nontrivial normal closed subgroup $N \subseteq G$, we conclude that $C = \{1\}$.

Suppose that $N \cap G_1 \neq \{1\}$. Then since G_1 is internally indecomposable, it holds that $C \cap G_1 \subseteq C_1 = \{1\}$. If $C \neq \{1\}$, then since G_2 is internally indecomposable, it holds that

$$\{1\} \neq N \subseteq Z_G(C) \subseteq G_1.$$

However, this contradicts Claim 1.11.A [in the case where $H = C$]. Thus, we conclude that $C = \{1\}$. This completes the proof of assertion (i).

Next, we verify assertion (ii). Recall that G_1 is center-free. Then, if ρ is injective, then G is also center-free. Thus, we may assume without loss of generality that G is center-free. Next, we verify the following assertion:

> Claim 1.11.B: Let $H \subseteq G$ be a nontrivial normal closed subgroup. Then $H \cap G_1 \neq \{1\}$.

Indeed, since $H \neq \{1\}$, and $Z(G) = \{1\}$, there exist elements $g \in G$, $h \in H$ such that $1 \neq x \stackrel{\text{def}}{=} g \cdot h \cdot g^{-1} \cdot h^{-1} \in G$. Fix such elements. Then since G_2 is abelian, the image of x via the surjection $G \twoheadrightarrow G_2$ is trivial. Moreover, since $H \subseteq G$ is normal, it holds that $x \in H$. In particular, we have $x \in H \cap G_1 \neq \{1\}$. This completes the proof of Claim 1.11.B.

Then, by applying Claim 1.11.B to the nontrivial normal closed subgroup $N \subseteq G$, we conclude that $N \cap G_1 \neq \{1\}$. Thus, since G_1 is internal indecomposability, it holds that $C \cap G_1 \subseteq C_1 = \{1\}$. Finally, it follows from Claim 1.11.B [in the case where $H = C$] that $C = \{1\}$. This completes the proof of assertion (ii), hence of Proposition 1.11. $\square$

2 Internal indecomposability of the absolute Galois groups

In the present section, we prove that the absolute Galois groups of

- Henselian discrete valuation fields with positive characteristic residue fields and
- Hilbertian fields

are strongly internally indecomposable [cf. Definition 1.1, (v)].

Let p be a prime number. The following lemma is well-known and elementary.

Lemma 2.1 ([8], Lemma 3.1)**.** *Let K be a Henselian discrete valuation field. Write $\widehat{K}$ for the completion of K; $f : G_{\widehat{K}} \to G_K$ for the natural outer homomorphism determined by the natural injection $K \hookrightarrow \widehat{K}$. Then f is bijective.*

Let us recall a result concerning the slimness of almost pro-p-maximal quotients of the absolute Galois groups of Henselian discrete valuation fields of characteristic p.

Proposition 2.2 ([8], Theorem 2.10)**.** *Let K be a Henselian discrete valuation field of characteristic p. Then G_K, as well as any almost pro-p-maximal quotient of G_K, is slim.*

The following result may be regarded as a generalization of Proposition 2.2 [cf. Remark 1.1.1].

Theorem 2.3. *Let K be a Henselian discrete valuation field of characteristic p; $N \subseteq G_K$ a normal open subgroup. Then G_K, as well as the almost pro-p-maximal quotient*

$$(G_K)_N \stackrel{\text{def}}{=} G_K/\text{Ker}(N \twoheadrightarrow N^p)$$

associated to N, is strongly internally indecomposable.

Proof. Note that $N^p \subseteq (G_K)_N$ is an open subgroup. Recall that $(G_K)_N$ is slim [cf. Proposition 2.2], and N^p is a free pro-p group of infinite rank [cf. [13], Proposition 6.1.7; Lemma 2.1]. Then it follows immediately from Propositions 1.5, 1.7, that $(G_K)_N$ is strongly internally indecomposable. Moreover, by varying N, we conclude that G_K is strongly internally indecomposable [cf. Proposition 1.8]. This completes the proof of Theorem 2.3. $\square$

Next, we recall the following well-known fact [cf. [2], Chapter III, §5; [16]]:

Theorem 2.4. *Let K be a mixed characteristic complete discrete valuation field such that the residue field of K is perfect and of characteristic p. Then the "field of norms"*

$$N(K_{p^\infty}/K) \stackrel{\text{def}}{=} \varprojlim_{i \geq 1} K(\zeta_{p^i})$$

— where the transition maps are the norm maps — satisfies the following conditions:

(a) $N(K_{p^\infty}/K)$ admits a natural structure of field [cf. [2], Chapter III, §5, (5.5), Theorem].

(b) $N(K_{p^\infty}/K)$ is isomorphic to $k((t))$, where k denotes the residue field of the [Henselian] valuation field K_{p^∞} [cf. [2], Chapter III, §5, (5.5), Theorem].

(c) $G_{K_{p^\infty}}$ is isomorphic to $G_{k((t))}$ [cf. [2], Chapter III, §5, (5.7), Theorem].

Also, let us recall a result concerning the slimness of almost pro-p-maximal quotients of the absolute Galois groups of abelian extensions of the fields of fractions of mixed characteristic Noetherian local domains.

Proposition 2.5 ([8], Theorem 2.8, (i), (ii))**.** *Let A_0 be a mixed characteristic Noetherian local domain of residue characteristic p. Write K_0 for the field of fractions of A_0. Let $K_0 \subseteq K$ $(\subseteq \overline{K})$ be an abelian extension. Then G_K is slim. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of G_K is slim.*

Theorem 2.6. *Let K be a mixed characteristic Henselian discrete valuation field of residue characteristic p. Then G_K and $G_{K_{p^\infty}}$, as well as any almost pro-p-maximal quotient of $G_{K_{p^\infty}}$, are strongly internally indecomposable. Moreover, if $\zeta_p \in K$, then any almost pro-p-maximal quotient of G_K is strongly internally indecomposable.*

Proof. First, it follows immediately from Proposition 1.7, together with [the first portion of] Proposition 2.5, that we may assume without loss of generality that

$$\zeta_p \in K.$$

Moreover, it follows from Propositions 1.7, 1.8, together with [the second portion of] Proposition 2.5, that it suffices to prove that G^p_K and $G^p_{K_{p^\infty}}$ are strongly internally indecomposable. Write k for the residue field of K.

Next, we verify the following assertion:

> Claim 2.6.A: Suppose that k is perfect. Then $G^p_{K_{p^\infty}}$ is strongly internally indecomposable.

Indeed, Claim 2.6.A follows immediately from Theorems 2.3, 2.4, together with Lemma 2.1.

Next, we verify the following assertion:

> Claim 2.6.B: $G^p_{K_{p^\infty}}$ is strongly internally indecomposable.

Indeed, let $\{t_i \ (i \in I)\}$ be a p-basis of k; $\tilde{t}_i \in K$ a lifting of t_i. For each $j \in \mathbb{N}$, let $\tilde{t}_{i,j} \in \overline{K}$ be a p^j-th root of $\tilde{t}_i \in K$ such that $\tilde{t}^p_{i,j} = \tilde{t}_{i,j-1}$, where $\tilde{t}_{i,0} \overset{\mathrm{def}}{=} \tilde{t}_i$. Write

$$L \ (\subseteq \overline{K})$$

for the field obtained by adjoining the elements $\{\tilde{t}_{i,j} \ ((i,j) \in I \times \mathbb{N})\}$ to K. Then L is a mixed characteristic Henselian discrete valuation field such that the residue field of L is perfect and of characteristic p. Therefore, it follows from Claim 2.6.A that $G^p_{L_{p^\infty}}$ $(\subseteq G^p_{K_{p^\infty}})$ is strongly internally indecomposable. On the other hand, we note that

- $G^p_{K_{p^\infty}}$ is slim [cf. [the second portion of] Proposition 2.5];
- $\mathrm{Gal}(L_{p^\infty}/K_{p^\infty})$ is abelian.

Now we apply Proposition 1.11, (ii), to the present situation, by taking "G" (respectively, "G_1"; "G_2") to be $G^p_{K_{p^\infty}}$ (respectively, $G^p_{L_{p^\infty}}$; $\mathrm{Gal}(L_{p^\infty}/K_{p^\infty})$). Thus, we conclude that $G^p_{K_{p^\infty}}$ is strongly internally indecomposable. This completes the proof of Claim 2.6.B.

Finally, we note that

- G_K^p is slim [cf. [the second portion of] Proposition 2.5];
- $\mathrm{Gal}(K_{p^\infty}/K)$ is isomorphic to $\mathbb{Z}_p$ [cf. the fact that $\zeta_p \in K$].

Now we apply Proposition 1.11, (ii), to the present situation, by taking "G" (respectively, "G_1"; "G_2") to be G_K^p (respectively, $G_{K_{p^\infty}}^p$; $\mathrm{Gal}(K_{p^\infty}/K)$) [cf. Claim 2.6.B]. Thus, we conclude that G_K^p is strongly internally indecomposable. This completes the proof of Theorem 2.6. □

Remark 2.6.1. It is natural to pose the following questions:

> Question 1: Is the absolute Galois group of any discrete valuation field with a positive characteristic residue field strongly internally indecomposable?
>
> Question 2: More generally, is the absolute Galois group of any subfield of a discrete valuation field with a positive characteristic residue field strongly internally indecomposable?

However, at the time of writing the present paper, the authors do not know whether the answer to each question is affirmative or not.

Next, we review the definition of higher local fields.

Definition 2.7 ([1], Chapter I, §1.1)**.** Let K be a field; $d \in \mathbb{N}$.

(i) A structure of *local field of dimension* d on K is a sequence of complete discrete valuation fields $K^{(d)} \overset{\mathrm{def}}{=} K, K^{(d-1)}, \ldots, K^{(0)}$ such that

- $K^{(0)}$ is a perfect field;
- for each integer $0 \leq i \leq d-1$, $K^{(i)}$ is the residue field of the complete discrete valuation field $K^{(i+1)}$.

(ii) We shall say that K is a *higher local field* if K admits a structure of local field of some positive dimension. In the remainder of the present paper, for each higher local field, we fix a structure of local field of some positive dimension.

Definition 2.8. Let K be a field. Then we shall say that K is *stably* μ_{p^∞}*-finite* if, for every finite extension field M of K, the group of p-power roots of unity $\in M$ is finite.

Here, let us recall a result concerning the slimness of the absolute Galois groups of higher local fields.

Proposition 2.9 ([8], Corollary 2.11, (iii))**.** *Let K be a higher local field. Suppose that* $\mathrm{char}(K^{(0)}) \neq 0$*, and $K^{(0)}$ is a stably μ_{l^∞}-finite field for any prime number l. Then G_K is slim. In particular, if $K^{(0)}$ is finite, then G_K is slim.*

Corollary 2.10. *Let K be a higher local field. Then the following hold:*

(i) Suppose that the residue characteristic of K is p. Then G_K is strongly internally indecomposable. Moreover, if $\zeta_p \in K$ in the case where $\mathrm{char}(K) = 0$*, then any almost pro-p-maximal quotient of G_K is strongly internally indecomposable.*

(ii) Suppose that $\mathrm{char}(K^{(0)}) \neq 0$*, and $K^{(0)}$ is a stably μ_{l^∞}-finite field for any prime number l. Then G_K is strongly indecomposable. In particular, if $K^{(0)}$ is finite, then G_K is strongly indecomposable.*

Proof. Assertion (i) follows immediately from Theorems 2.3, 2.6.

Next, we verify assertion (ii). It follows immediately from assertion (i) that we may assume without loss of generality that the residue characteristic of K is 0. Since every finite extension of K is a higher local field of residue characteristic 0, it suffices to prove that G_K is indecomposable. Suppose that there exist normal closed subgroups $H_1 \subseteq G_K$ and $H_2 \subseteq G_K$ such that

$$G_K = H_1 \times H_2.$$

Write $i \in \mathbb{N}$ for the positive integer such that $\mathrm{char}(K^{(i+1)}) > 0$. Recall from Cohen's structure theorem that

$$K \cong K^{(i)}((t_1)) \cdots ((t_m)).$$

Then we have an exact sequence of profinite groups

$$1 \longrightarrow \widehat{\mathbb{Z}}(1)^{\oplus m} \longrightarrow G_K \longrightarrow G_{K^{(i)}} \longrightarrow 1,$$

where "(1)" denotes the Tate twist. Here, we note that $G_{K^{(i)}}$ is internally indecomposable [cf. (i)]. In particular, it holds that $H_1 \subseteq \widehat{\mathbb{Z}}(1)^{\oplus m}$ or $H_2 \subseteq \widehat{\mathbb{Z}}(1)^{\oplus m}$. We may assume without loss of generality that $H_1 \subseteq \widehat{\mathbb{Z}}(1)^{\oplus m}$. Then since $G_K = H_1 \times H_2$, and H_1 is abelian, it holds that $H_1 \subseteq Z(G_K)$. Thus, since $Z(G_K) = \{1\}$ [cf. Proposition 2.9], we conclude that $H_1 = \{1\}$. This completes the proof of assertion (ii), hence of Corollary 2.10. □

Remark 2.10.1. Let K be a field of characteristic 0. Then we have an exact sequence of profinite groups

$$1 \longrightarrow \widehat{\mathbb{Z}}(1) \longrightarrow G_{K((t))} \longrightarrow G_K \longrightarrow 1.$$

Note that $\widehat{\mathbb{Z}}(1) \subseteq G_{K((t))}$ is a normal closed subgroup, and $\widehat{\mathbb{Z}}(1)$ is not center-free. Thus, we conclude that $G_{K((t))}$ is not internally indecomposable.

Next, let us recall a result concerning the slimness of the absolute Galois groups of Hilbertian fields.

Proposition 2.11 ([7], Theorem 2.1)**.** *Let K be a Hilbertian field. Then G_K is slim and strongly indecomposable.*

The following result may be regarded as a generalization of Proposition 2.11 [cf. Remarks 1.1.1; 1.1.2, (ii)].

Theorem 2.12. *Let K be a Hilbertian field. Then G_K is strongly internally indecomposable.*

Proof. Since every finite separable extension of K is Hilbertian [cf. [3], Corollary 12.2.3], it suffices to prove that G_K is internally indecomposable. Let $N \subseteq G_K$ be a nontrivial normal closed subgroup. Write $C \stackrel{\text{def}}{=} Z_G(N)$. Then it follows immediately from the various definitions involved that

$$C \cap N = Z(N) \subseteq G_K$$

is an abelian normal closed subgroup. Thus, by applying [3], Proposition 16.11.6, we conclude that $C \cap N = \{1\}$, hence that $C \cdot N = C \times N \subseteq G_K$.

Next, we recall that G_K is slim [cf. Proposition 2.11]. Since $N \subseteq G_K$ is a nontrivial normal closed subgroup, it follows immediately from Lemma 1.6 that N is infinite. Let $N^\dagger \subsetneq N$ be a proper nontrivial normal open subgroup. Then $C \times N^\dagger \subsetneq C \times N = C \cdot N$ is a proper normal open subgroup. Thus, by applying Weissauer's theorem [cf. [3], Theorem 13.9.1, (b)], we conclude that $C \times N^\dagger$ is isomorphic to the absolute Galois group of a Hilbertian field. In particular, $C \times N^\dagger$ is indecomposable [cf. Proposition 2.11; [3], Corollary 13.8.4]. Since $N^\dagger \neq \{1\}$, this implies that $C = \{1\}$. Thus, we conclude that G_K is internally indecomposable [cf. Proposition 1.2]. This completes the proof of Theorem 2.12. □

Remark 2.12.1. It is well-known that the following hold:

(i) The field of fractions of an arbitrary integral domain that is finitely generated over $\mathbb{Z}$ is Hilbertian [cf. [3], Proposition 13.4.1].

(ii) Finitely generated transcendental extension field of an arbitrary field is Hilbertian [cf. [3], Proposition 13.4.1].

(iii) The field of fractions of an arbitrary Noetherian integral domain of dimension ≥ 2 is Hilbertian [cf. [3], Theorem 15.4.6; [6], p296, Mori-Nagata's integral closure theorem].

In particular, it follows from Theorem 2.12 that the absolute Galois groups of the above fields are strongly internally indecomposable.

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References

[1] I. Fesenko and M. Kurihara (eds.), *Invitation to higher local fields*, *Geometry and Topology monographs vol. 3, Geometry and Topology Publications. International Press* (2000).

[2] I. Fesenko and S. Vostokov, *Local fields and their extensions (Second edition)*, *Providence. R. I, Translations of mathematical monographs* **121** (2002).

[3] M. Fried and M. Jarden, *Field arithmetic (Second edition)*, *Ergebnisse der Mathematik und ihrer Grenzgebiete 3. Folge, A Series of Modern Surveys in Mathematics* **11**, Springer-Verlag (2005).

[4] A. Grothendieck and M. Raynaud, *Revêtements étales et groupe fondamental* (SGA1), *Lecture Notes in Math.* **224** (1971), Springer-Verlag.

[5] Y. Hoshi, The absolute anabelian geometry of quasi-tripods. *Kyoto J. Math.* **62** (2022), pp. 179–224.

[6] H. Matsumura, *Commutative algebra (Second edition)*, *Mathematics Lecture Note Series* **56**, Benjamin/Cummings Publishing Company (1980).

[7] A. Minamide, Indecomposability of various profinite groups arising from hyperbolic curves, *Okayama Math. J.* **60** (2018), pp.175–208.

[8] A. Minamide and S. Tsujimura, Anabelian group-theoretic properties of the absolute Galois groups of discrete valuation fields, *J. Number Theory*

(2022), available at the following URL
`https://doi.org/10.1016/j.jnt.2021.12.006`

[9] S. Mochizuki, The local pro-p anabelian geometry of curves, *Invent. Math.* **138** (1999), pp. 319–423.

[10] S. Mochizuki, Topics in absolute anabelian geometry I: Generalities, *J. Math. Sci. Univ. Tokyo* **19** (2012), pp. 139–242.

[11] S. Mochizuki, Inter-universal Teichmüller theory I: Construction of Hodge theaters, *Publ. Res. Inst. Math. Sci.* **57** (2021), pp. 3–207.

[12] S. Mochizuki and A. Tamagawa, The algebraic and anabelian geometry of configuration spaces, *Hokkaido Math. J.* **37** (2008), pp. 75–131.

[13] J. Neukirch, A. Schmidt, K. Wingberg, *Cohomology of number fields*, *Grundlehren der Mathematischen Wissenschaften* **323**, Springer-Verlag (2000).

[14] L. Ribes and P. Zaleskii, *Profinite groups*, *Ergebnisse der Mathematik und ihrer Grenzgebiete* **3**, Springer-Verlag (2000).

[15] A. Tamagawa, The Grothendieck conjecture for affine curves, *Compositio Math.* **109** (1997), pp. 135–194.

[16] J.-P. Wintenberger, Le corps des normes de certaines extensions infinies des corps locaux; applications, *Ann. Sci. École Norm. Sup.* **16** (1983), pp. 59–89.

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